THE APPLICATION OF MONETARY POTENTIAL THEORY IN OPTION PRICING PROBLEMS

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ABSTRACT. Options are the most basic financial derivatives. Using the theory of monetary potentials can solve the American and European (call and put) option pricing problems. In particular, the American putting premium $Z$ can be expressed as a single money potential with density $\theta$ and distributed on the American put option optimal exercise boundary $\Gamma$, i.e., as a money influence function of money sources with intensity $\theta$ and acting continuously along $\Gamma$. From the financial point of view, these algorithms and solutions are more natural and intuitive. From this expression, we can see that the better the put situation in the stock market, the higher the price of an American put premium. In this case, a buyer of an American put option will reap a larger profit than a buyer of a European put option in the same conditions. A longer exercise period results in more selection for a buyer of an American put option, which is, in turn, greater than the selection available to the expansive American put premium. Whenever the price of an American put premium is higher, an exercise of the American put option by the buyer will result in maximum profit.

Keywords: Option Pricing; Black-Scholes Model; Money Potential; Premium

1. Introduction. In 1973, professor Fischer Black of the University of Chicago and professor Myron Scholes of the Massachusetts Institute of Technology in “Journal of Political Economy” published an article entitled “The Pricing of Options and Corporate Liabilities” (Black and Scholes, 1973), laid the theoretical basis of option pricing, creating a new investment area of finance. They shared the 1997 Nobel Prize in Economics. They wrote: "If options are correctly priced in the market, it should not be possible to make sure profits by creating portfolios of long and short positions in options and their underlying stocks."

Black-Scholes equation (BS equation or BS model) is a second order stochastic parabolic partial differential equation. The trading conditions of options can be considered as the initial conditions or boundary conditions of this equation (Black and Scholes, 1973; Jiang, 2008 and Sun, 2008). By means of some mathematical transformations, this BS equation can be converted into a partial differential equation of heat conduction. In particular, the pricing problem of the American put premium forms a free boundary problem of this heat conduction equation, which kind of problem, the author of this paper had discussed 45 years ago (Wang, 1965 and 1980).
With China’s financial reform in depth, in this century, Chinese scholars have strengthened the study on the theory of option pricing. Jiang Li Shang and his students are the most prominent, and they are also made a number of China’s financial (derivative) application of BS equation. (Jiang et al., 2008).

The author of this paper initiated the monetary potential theory in 2006, and succeeded in characterization of e-finance activities and transient characteristics of clicks. Determining the basic properties of various monetary potentials, especially after the introduction of direct value concept for single money potential and dual money potential, using their “jump ” formula solved many monetary circulation problems with different initial conditions and boundary conditions for the heat conduction equation (according to their financial significance, also known as the money circulation equation). (Wang, 1980, 2006; Wang et al., 2009).

Using the monetary potential theory can calculate the European option and American option pricing and other options. From the financial point of view, these algorithms and solutions are more natural and intuitive, while for American put premium it can further found some important characteristics of financial nature: when the better the put situation in the stock market, then the higher the price of American put premium. In this case a buyer of American put option will get more payoff than a buyer of European put option in the same conditions. The longer the exercise period, the more selection any buyer of American put option has than that of the expansive American put premium. When the price of American put premium is the highest, if the buyer of American put option exercises, he can get the maximum payoff.

2. The Basic Concepts of Options. Options are the most basic of financial derivatives. Their price ultimately depends on the price of another underlying asset. Option is a selection right, the option holder offers his price to buy or sell the underlying asset in the appointed time, but assuming the no obligation to buy or sell. Call option and put option are two common options. Option contract entered on the date of signing, option expiry date is the due date stipulated in the contract. According to different exercise freedom in the life of the option, we can distinguish American option (can be exercised in any day before the contract expiry date) and European option (exercised only in the contract expiry date).

European option is firstly introduced by the London Futures Exchange; American option is firstly introduced by the Chicago Board of Trade; China's capital market is only 20 years old, and it has not yet launched options trading.

3. BS Equation (Model) and Its Simplified. Without loss of generality, we discuss a kind of (American or European) option, and take a stock (share*) as the underlying asset. In the stock market with some assumptions (such as the no-arbitrage and risk neutral, and shares without dividends, etc.), the option (price) satisfies the following second order stochastic parabolic partial differential equation (Black and Scholes, 1973)

\[
\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS \frac{\partial F}{\partial S} - rF = 0
\]  

among them

\[F(S,t)\] the price of an option;
S = S(t): the underlying stock's price;

\( t \): the time (signing date \( t = 0 \), expiry date \( t = T \));

\( \sigma \): the underlying stock price volatility; it is a constant;

\( r \): the (risk-free) interest rate; it is a constant.

Equation (1) is called the BS equation (or model) in the plane \((S,t)\) (in the coordinate system \((S,t)\)), which can be transformed into a heat conduction equation.

We make replacement of variables in (1) and (1) will be simplified. To make dependent variable replacement firstly in (1).

Let \( H(S,t) = F(S,t) \exp(-rt) \)

The equation (1) will be changed to:

\[
(\partial H / \partial t) \exp(rt) + rH \exp(rt) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} \exp(rt) + rS \frac{\partial H}{\partial S} \exp(rt) - rH \exp(rt) = 0
\]

Or

\[
\frac{\partial H}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + rS \frac{\partial H}{\partial S} = 0;
\]

Do replace the stock price variable

\( y = \ln S \), \( H(S,t) = H(e^y, t) = K(y, t) \);

equation (1) is into

\[
\frac{\partial K(y, t)}{\partial t} + \frac{1}{2} \sigma^2 e^{2y} \frac{\partial^2 K(y, t)}{\partial y^2} + r \frac{\partial K(y, t)}{\partial y} = 0.
\]

For eliminating the first order partial differential of \( K(y, t) \) to \( y \), do replacement contacting the stock price and the time independent variables.

\( w = y + rt, \ K(y, t) = K(w - rt, t) = J(w, t) \).

Variation of the above equation is

\[
\frac{\partial J(w, t)}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 J(w, t)}{\partial w^2} = 0;
\]

To make the above equation into an equation with unit coefficients, the final set is

\( x = \sqrt{\frac{2}{\sigma}} w, J(w, t) = J(\frac{\sigma}{\sqrt{2}} x, t) = M(x, t) \)

The equation mentioned above is into an equation with unit coefficients:

\[
M(x, t)_{xx} + M(x, t)_{t} = \frac{\partial^2 M(x, t)}{\partial x^2} + \frac{\partial M(x, t)}{\partial t} = 0.
\]

This is a (in the coordinate system \((x, t)\)) reverse heat conduction equation with unit coefficients.

Finally, do the time change \( T - t = \tau \), \( M(x, t) = M(x, T - \tau) = Z(x, \tau) \).

The equation cited above is transformed (in the plane \((x, \tau)\), in the coordinate system \((x, \tau)\)) into a positive heat conduction equation with unit coefficients. This is the (simplified) BS equation (model):

\[
Z(x, \tau)_{xx} - Z(x, \tau)_\tau = \frac{\partial^2 Z(x, \tau)}{\partial x^2} - \frac{\partial Z(x, \tau)}{\partial \tau} = 0. \tag{2}
\]

This is the BS equation in the plane \((x, \tau)\).
4. Initial Condition and Boundary Condition in (European or American) Option Pricing Problems.

4.1. Initial Condition in European and American Option Pricing Problems.

(1) The exercise conditions for European option and American option in the coordinate system \( (S, t) \) can be expressed in terms of the boundary conditions of equation (1).

In the plane \( (S, t) \), when \( t = T \):

European call option (call right) exercise conditions are

\[
F(S, T) = \max(0, S(T) - L)
\]

where \( L \) is the agreed price (strike price) of exercise in the option contract.

If at \( t = T \) the \( S(T) \) is bigger (higher) than \( L \) and the option holder exercises his option and buys the share (underlying asset) with agreed price \( L \), he will get payoff \( S(T) - L \), so \( F(S, T) = S(T) - L \);

If at \( t = T \) the \( S(T) \) is smaller (lower) than \( L \) and the option holder does not exercise his option and does not buy the share (underlying asset), he will not get any payoff, so \( F(S, T) = 0 \);

European put option (put right) exercise conditions are

\[
F(S, T) = \max(0, L - S(T))
\]

Where \( L \) is the agreed price (strike price) of exercise in the option contract.

If at \( t = T \) the \( L \) is bigger (higher) than \( S(T) \) and the option holder exercises his option and sells his share (underlying asset) with agreed price \( L \), he will get payoff \( L - S(T) \), so \( F(S, T) = L - S(T) \);

If at \( t = T \) the \( L \) is smaller (lower) than \( S(T) \) and the option holder does not exercise his option and does not sell his share (underlying asset), he will not get any payoff, so \( F(S, T) = 0 \);

The exercise conditions of American option are the same as shown above, but the option holder can exercise option in any day before the termination. Which day is the best will be detailed in the following discussion.

(2) The exercise conditions for European option and American option in the coordinate system \( (S, t) \) can be expressed in terms of the initial conditions of equation (2) in the coordinate system \( (x, \tau) \).

Utilizing the variable mentioned above transformations, the relationship between \( F(s, t) \) and \( Z(x, \tau) \) can be identified.

Because \( H(S, t) = F(S, t) \exp(-rt), H(S, t) = H(e^r, t) = K(y, t) \),

\[
K(y, t) = K(w - rt, t) = J(w, t), J(w, t) = J(\frac{\sigma}{\sqrt{2}}, x, t) = M(x, t),
\]

therefore

\[
F(S, t) = e^{rt} K(y, t) = e^{rt} J(w, t) = e^{rt} M(x, t) = e^{rt} Z(x, \tau) = e^{r(T-\tau)} Z(x, \tau).
\]

In the transformations above, we obviously see that

\[
\begin{align*}
S &= e^y = e^{w-rt} = e^{\frac{\sigma}{\sqrt{2}}e^{r-t}}, \quad x = \frac{\sqrt{2}}{\sigma} \left[ \ln S + rt \right].
\end{align*}
\]

Thus, (3) is:
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\[ Z(x,0) = e^{-rT} F(S,T) = e^{-rT} \max(0, S - L) = e^{-rT} \max(0, (e^{\frac{\sigma x}{2} - rT} - L)) = f(L, x, T). \quad (5) \]

(4) is converted to

\[ Z(x,0) = e^{-rT} \max(0, (L - e^{\frac{\sigma x}{2} - rT})) = g(L, x, T). \quad (6) \]

(5) and (6) can be regarded as the initial conditions of the equation (2) in the coordinate system \((x, \tau)\).

(3) For the European (call or put) option pricing problem, it need to solve the equation (1) with the initial conditions (3) (or (4)) or to solve the equation (2) with the initial conditions (5) (or (6)).

As is well known on the American call option, to early exercise this kind of option before the expiry day is meaningless (Jiang, 2008). So, on the American call option, the same should solve the equation (1) with the condition (3) (or the equation (2) with the condition (5)).

As is well known too, the European (call or put) option pricing problem and the American call option pricing problem (problem (2, 5) or (2, 6)) can be all solved by means of the plane money potential. (Wang et al., 2009).

4.2. Boundary Conditions for American Put Option Pricing Problem.

When discussing the American put option, the due date and the exercise date have great significance. Clearly, if only the put option buyers get more payoffs, they will exercise option before the expiry date. According to the market principle that no arbitrage can be proved the existence inside the American put option optimal exercise boundary \(\Gamma\) (its curve equation is \(S = k(t), 0 \leq t \leq T\): it also known as a free boundary. To exercise option at this boundary (in the appropriate time and price determined by this boundary) can get greater benefits than to exercise option in the expiry date. Moreover, the partial derivative of option to the stock price at this boundary takes value of -1 (Wilmott et al., 1995; Zhu, 2006).

When the American put option pricing is concerned, except the equation (1) and initial condition (4), the solution should yet meet the (moving) boundary conditions (in the plane \((S, t)\))

\[ [F(S, t)|S = k(t)] = L - k(t) \quad (7) \]

\[ [\partial F(S, t)/\partial S|S = k(t)] = -1 \quad (8) \]

The solutions of the problem ((1), (4), (7), (8)) are the option price function \(F(S, t)\) and the optimal exercise boundary function \(k(t)\), which is called the optimal exercise price.

In the plane \((S, t)\) constructing the fundamental solution of the BS equation (1) and decomposing the problem ((1),(4),(7),(8)) to two parts and using a nonlinear integral equation, this problem can be solved, but the final solution takes only a non analytical form (Jiang,2008)

In the plane \((x, \tau)\), because \(x = \frac{\sqrt{2}}{\sigma} [\ln S + rt]\), therefore, the optimal American put option exercise boundary \(\Gamma\) can be expressed as \(x = X(\tau)\),and
\[ X(\tau) = \frac{\sqrt{2}}{\sigma} \left[ \ln k(T - \tau) + r(T - \tau) \right] \]  

\[ k(t) = \exp\left[ \frac{\sigma}{\sqrt{2}} X(\tau) - rt \right] \]  

From these forms we can see that when \( k(t) \) is known then \( X(\tau) \) is known also and vice versa.

When we write the condition (7) in the plane \((x, \tau)\), we should pay attention to \( F(S,t) = e^{(T-t)} Z(x, \tau) \), therefore

\[ [Z(x, \tau) \big| x = X(\tau)] = [e^{(T-t)} F(S,t) \big| S = k(t)] = e^{(T-t)}[L - k(T - \tau)] = A(\tau). \]  

When we write the condition (8) in the plane \((x, \tau)\), we should note

\[ \frac{\partial F}{\partial S} = e^{(T-t)} \frac{\partial Z}{\partial x} \]  

therefore, on \( \Gamma \)

\[ \frac{\partial Z}{\partial x} = e^{(T-t)} \frac{\partial F}{\partial S} \left[ \frac{\sigma S}{\sqrt{2}} \right] = e^{(T-t)} \frac{\sigma S}{\sqrt{2}} = e^{\frac{\sigma}{\sqrt{2}}} = D(x, \tau) \]

Condition (8) can be written as:

\[ [\frac{\partial Z}{\partial x} \big| x = X(\tau)] = [D(x, \tau) \big| x = X(\tau)] = B(\tau). \]

4.3. American Put Option Pricing Problem Formulation (in the Plane \((x, \tau)\)). In the plane \((x, \tau)\) solving the American put option pricing problem ((2), (6), (9), (10)) needs to find the option price and the optimal exercise boundary. This American put option pricing problem is the one with dual boundary conditions ((9) and (10)) and single initial condition (6). The unknown boundary and the duality of the boundary conditions are the problems to be solved difficulty.

We will use the monetary potential theory to solve this problem ((2), (6), (9), (10)). Founding \( Z(x, \tau) \) and \( X(\tau) \) according to the relationship between variables, we can find finally \( F(S,t) \) and \( k(t) \) to satisfy equation (1), initial condition (4), boundary conditions (7) and boundary conditions (8).

5. American Put Premium Pricing Problem (in the Plane \((x, \tau)\)). The solving path of the American put option problem ((2), (6), (9), (10)) in the plane \((x, \tau)\) is: in view of the linear nature of equation (2), based on the principle of superposition, firstly using European put option problem solving method (Jiang, 2008, Wang et al. 2009), the condition (6) will be transformed into the "zero initial value" condition. At this time, the American put option "auxiliary solution \( Y(x, \tau) \)" satisfies equation (2) and the initial condition (6) will be found. The "auxiliary solution \( Y(x, \tau) \)" meets

\[ Y(x, \tau)_{xx} - Y(x, \tau)_{x} = 0 \]

And the initial condition (6): \( Y(x,0) = g(L, x, T) \).

Finding the changes of appropriate boundary conditions (9) and (10) (note the new boundary conditions as conditions (A) and (B)), the American put option problem with
"non-zero initial value" is changed into a new American put option problem with the "zero initial value".

In this "zero initial value" American put option problem, also known as the American put option problem by paying the premium issue, paying the premium is due to the possibility to early exercise option (earlier than maturity); this problem is referred as the American put premium (by payment of the premium) problem. We will write down this "zero initial value" problem of American put premium.

American put premium

\[ W(x, \tau) = Z(x, \tau) - Y(x, \tau) \]

meets the heat conduction equation with unit coefficients

\[ W(x, \tau)_{xx} - W(x, \tau)_{\tau} = 0, \quad 0 \leq \tau \leq T, \, X(\tau) \leq x < \infty, \quad (11) \]

boundary conditions

\[
\begin{cases}
W(X(\tau), \tau) = \phi(\tau) = A(\tau) - Y(X(\tau), \tau), 0 \leq \tau \leq T, (A) \\
W_{\tau}(X(\tau), \tau) = \psi(\tau) = B(\tau) - Y_{\tau}(X(\tau), \tau), 0 \leq \tau \leq T, (B)
\end{cases}
\]

and zero initial condition

\[ W(x, 0) = 0, \, X(0) \leq x < \infty. \quad (13) \]

The new boundary conditions \( \phi(\tau) \) and \( \psi(\tau) \) in (12) are obtained on base of the original boundary conditions (9), (10) and the corresponding values on the boundary \( \Gamma \) of the auxiliary solution \( Y(x, \tau) \). (And, \( \phi(0) = 0 \), it said that the initial condition (13) and the first boundary condition in (12) are coordinated.)

6. To Solve the American Put Premium Pricing Problem Using the Monetary Potential Theory. According to the monetary potential theory Wang (2006) (please see the appendix), the solution of the problem ((11), (12), (13)) can be written as a single money potential distributed on \( \Gamma \):

\[ W(x, \tau) = V_{\Gamma}(\theta) = \int_0^{\tau} \theta(\eta)G(x, \tau; X(\eta), \eta)\,d\eta \quad (14) \]

here

\[ G(x, \tau; \xi, \eta) = \frac{1}{2\sqrt{\pi(\tau - \eta)}} \exp\left[-\frac{(x - \xi)^2}{4(\tau - \eta)}\right] \quad (15) \]

is the money influence function (i.e., the fundamental solution of the heat conduction equation (2) (or 11) with unit coefficients). \( \theta \) is the density of single money potential \( V_{\Gamma}(\theta) \) (14). From the monetary potential theory we know that the single money potential \( V_{\Gamma}(\theta) \) meets the equation (11) and the "zero initial value" condition (13) (Wang, 1965; 1980).

According to the “jump” formula for the direct value of partial derivatives of single money potential to space variable, to satisfy the second condition (B) in condition (12), on the \( \Gamma \) for the density \( \theta \) of the single money potential, it should be validated a linear Volterra integral equation of second type:

\[ \psi(\tau) = \theta(\tau) / 2 + \int_0^{\tau} \theta(\eta)K(X(\tau), \tau; X(\eta), \eta)\,d\eta \quad (16) \]
among
\[
\int_0^r \theta(\eta) K(X(\tau), \tau; X(\eta), \eta) d\eta = \int_0^r \theta(\eta) [\partial G(X(\tau), \tau; X(\eta), \eta) / \partial \xi] d\eta
\]
is the direct value on $\Gamma$ of the partial derivative to space variable of the single money potential $V_\Gamma'(\theta)$. (Wang, 1965; 1980; 2006) The integral equation (16) is
\[
\theta(\tau) = 2B(\tau) - \int_0^r \frac{\theta(\eta) (X(\tau) - X(\eta))}{2\sqrt{\pi}(\tau - \eta)} \exp\left[-\frac{(X(\tau) - X(\eta))^2}{4(\tau - \eta)}\right] d\eta - 2Y_\tau(X(\tau), \tau) \tag{17}
\]
As $X(\tau)$ is known, this is a linear integral equation for $\theta(\tau)$.

To satisfy the first condition (A) in conditions (12), an equation under this condition is written:
\[
W(X(\tau), \tau) = \varphi(\tau) = V_\Gamma'(\theta)|_{x = X(\tau)} \tag{18}
\]
Integral equation (18) is
\[
A(\tau) = Y(X(\tau), \tau) + \int_0^r \frac{\theta(\eta)}{2\sqrt{\pi}(\tau - \eta)} \exp\left[-\frac{(X(\tau) - X(\eta))^2}{4(\tau - \eta)}\right] d\eta \tag{19}
\]
As $\theta(\tau)$ is known, this is a nonlinear integral equation for $X(\tau)$.

Equation (17) and equation (19) compose a simultaneous nonlinear integral equation system of two unknown functions $X(\tau)$ and $\theta(\tau)$. $X(\tau)$ and $\theta(\tau)$ can be solved by this equation system. Substituting them into (14), we can get solution $W(x, \tau)$ of American put premium problem with "zero initial value"; re-considering superposition of European put option auxiliary solution $Y(x, \tau)$, we can ultimately get the solution $Z(x, \tau) = W(x, \tau) + Y(x, \tau)$ of American put option problem (2,6,9,10).

As the highly nonlinear nature, it is difficult to obtain the explicit analytical solution of (17) and (19).

However, there is extensive literature to research their approximate solution and the nature of optimal exercise boundary.

However, by means of expressions (17) and (19), we can get a number of qualitative information and then strengthen some knowledge of financial nature about the American put premium.

Using the tools provided in the monetary potential theory, we can also discuss other types of option pricing problems.

7. Some Financial Nature of American Put Premium. The equation (20) shows that in the plane $(x, \tau)$ the American put premium $W(x, \tau)$ can be expressed as a single money potential distributed on the optimal exercise boundary $\Gamma$ with the density $\theta$, i.e. as a money influence function of a money source with intensity $\theta$ and continuously acting at $\Gamma$.

\[
W(x, \tau) = \int_0^\tau \frac{\theta(\eta)}{2\sqrt{\pi}(\tau - \eta)} \exp\left[-\frac{(x - X(\eta))^2}{4(\tau - \eta)}\right] d\eta \tag{20}
\]

At this time, $\theta$ is known as the American put premium density. The relationship between the American put premium and its density is shown by equation (20). Clearly,
American put premium prices can be estimated with the maximum value of its density in the period of option exercise.

As is well known, the stock price and time plane \((x, \tau)(or(S, \tau))\) are divided by the optimal exercise boundary \(\Gamma\) into two parts \(\Sigma_1\) and \(\Sigma_2\). In the \(\Sigma_1\), the option buyer shall continue to hold option contract and in \(\Sigma_2\) the option holder shall terminate the option contract. Continuing to hold and the termination of possession are two different (economic) behaviors, which is separated (divided) by the boundary line \(\Gamma\). In other words, the option buyer does not act continuously through this line \(\Gamma\), while, on \(\Gamma\) buyer’s behavior reveals a "jump".

Similarly, as is well known (20) is the single money potential distributed on \(\Gamma\) with density \(\theta\), its partial derivatives to the space (stock price) variable also has a "jump" in the cross \(\Gamma\), that is, the difference of the limit values of this derivative on both sides of \(\Gamma\) which is precisely equal to the density \(\theta\) of this single money potential.

As a result, the change of American put premium \(W(x, \tau)\) on the stock market (that is, its partial derivatives to the price) is continuous in the both sides of optimal exercise boundary \(\Gamma\) and on \(\Gamma\) there is a "jump" shown. The value of this "jump" is just the density \(\theta\) of this American put premium. This "jump" is precisely corresponding with the "jump" of the option buyer’s behavior.

The equation (20) also shows that when bearish stock market is better, and the put situation is more clear, namely, the smaller one is \(x\) in (20) (and therefore also smaller is \(S\)), then obviously, the greater one is the corresponding value \(W\). That is to say: when the stock market is better put, the American put premium price will be higher. Therefore, the purchase of American put option can gain more income than that of European put option with the same conditions. So the purchase of American put option should be paying more premiums.

By (20) is also not difficult to see, for the same \(x\), the bigger \(T\) is, then the higher the corresponding value \(W(x, T)\). That is, the longer the period of option exercise, the more choices to get more revenue the option buyers have. Therefore, it meets more buyers’ premium.

The American put premium \(W(x, \tau)\) satisfies equation (2). According to the maximum principle for heat conduction equation, the maximum of \(W(x, \tau)\) is reached on the \(\Gamma\).

Let \(W(x_0, \tau_0) = \max W(x, \tau)\), then \(x_0 = X(\tau_0), (x_0, \tau_0) \in \Gamma\).

Obviously, in the time \(\tau_0\) and in the share price

\[
S = S_0 = \exp\left(\frac{\sigma}{\sqrt{2}} x_0 \right) - r(T - \tau_0) \quad \text{(corresponding to } x_0) \]

will be the highest American put premium price. If the American put option buyer exercises option at this time, he will receive the maximum benefit (payoff).

8. Similarity between the Money Circulation Problem and the Option Pricing Problem.

Stock (or other underlying assets) trading is as money putting and withdrawing. From stock market trading are derived options, from money putting and withdrawing is derived money circulation.
Stock price does not take negative, also the money values does. The stock market, like the money market, is all discussed with non negative time value: that, the money circulation problem and the option pricing problem have the same time domain of definition.

Microscopic randomness of stock trading is as microscopic randomness in money putting and withdrawing, then in the macro aspect they can be characterized and studied by means of the same methods and second-order stochastic parabolic partial differential equations.

The option pricing has initial value problem (such as the European option pricing problem), and money circulation also has the issue; the option pricing has boundary value problem (such as the American option pricing problem), the money circulation has boundary value problem, too.

The different economic and financial policies will affect the circulation of money; similarly, the different economic and financial policies will also affect option pricing. Option pricing and money circulation, in essence, are the financial activities and financial processes. The essence of consistency determines the consistency of their study methods. The monetary potential theory is a powerful tool to study the money circulation, naturally also the option pricing.

Many natural phenomena or processes have both "now", another "past" and "future" and their "now", "past" and "future" can be observed and measured simultaneously. The process of their change and evolution over time has rules to follow. Water flow process and some of the meteorological processes are like this. These phenomena or processes, mathematically, are often described and analyzed by hyperbolic partial differential equations.

For some natural phenomena, processes and some financial activities, we cannot simultaneously observe their "now", "past" and "future." For these phenomena, processes and activities, finding their change and evolution over time according to their "last" record and "now" situation, we may infer their "future". These phenomena and processes, mathematically, are often studied with the use of parabolic partial differential equations: you can use the parabolic partial differential equations from the "past" and "now" of some processes conclude its "future," to predict its "future."

Option pricing, in a sense, is a stock market prediction and forecasting.

The phenomena, processes and activity described by parabolic partial differential equations are irreversible processes in time order. It is impossible to repeat and reproduce this process. Molecular motions caused by heat conduction, thermal diffusion and thermal convection are such processes. Electronic money movement, particularly movement of bank notes, stock trading process, and option price change process are like the thermal motion of molecules so much. They have the same random micro; their macros can be used in parabolic partial differential equations.

Financial activities in human society are exactly such irreversible processes. It is impossible to make money once again flows and option pricing through various states in their development process in completely opposite chronological order.

In a parabolic partial differential equation, time variable and space variables are asymmetric. Roughly speaking, the one time derivative to time variable of financial activities (currency in circulation, option pricing and other financial process) is equivalent to two times derivative to space variables. Thus, in these activities, the time variable is more “valued” variable. In fact, in economic and financial activities, we often feel that time
is more important than space. "Time is life", "time is money," "time is efficiency" have become the mottos of economic and financial activities. (When we buy shares, futures, options and other financial products in the stocks market, the key is to seize the opportunity to sign orders and contracts, but not the place, the securities or the futures companies).

REFERENCES

Appendix.

Brief introduction of the monetary potential theory (Wang, 2006)

If at a moment \( t \) and in a place \( x \) we put (or withdrawn) money, then we say that at this moment \( t \) and in this place \( x \) there is a money source. The money amount put (or withdrawn) by some money source in a unit of time and in a unit of area around this source is called the intensity of this source.

Suppose that in the place \( x \) there is a put source of money, which continuously acts in the time interval \( 0 \leq t \leq T \). If in a unit of time this source continuously puts money \( f(t) \), then we say that in this place there is a continuously acting money source with the intensity \( f(t) \).

The money influence function for continuous from the moment \( t = 0 \) to the moment \( t \) acting of this money put source is here

\[
\int_{0}^{t} f(\tau)G(x, t, 0, \tau)d\tau = V[\mu] = V(x, t)
\]

acting of this money put source is here

\[
G(x, \tau; \xi, \eta) = \frac{1}{2\pi(\tau - \eta)} \exp\left[\frac{(x - \xi)^2}{4(\tau - \eta)}\right]
\]

is the money influence function (i.e., the fundamental solution of the heat conduction equation (2) with unit coefficients)

This integral \( V[\mu] \) is called the single money potential, \( \mu = f(t) \) is its density.

It is easy to test that when

\[
0 < t \leq T, x \neq 0
\]

this single money potential satisfies the heat conduction equation with unit coefficients.

\[
V_{x=0} = [V_x]_{x=0} = 0
\]

Obviously, We can define the direct value on the line \( x = 0 \) of single potential derivative to \( x \) and introduce its jump formula.

Jump formula for the derivative to \( x \) of the single money potential

\[
V_{x=0}^{+} = 0, t > 0
\]

\[
V_{x=0}^{-} = \frac{-\mu(t0)}{2}, t > 0
\]

here, \( (+) \) means that \( (x, t) \) tends to \( (0, t0) \) on the straight line \( (x = 0) \) from its right side; \( (-) \) means that \( (x, t) \) tends to \( (0, t0) \) on the straight line from its left side.