THE VARYING ELASTICITY PRODUCTION FUNCTION WITH FRONTIER EFFICIENCY: A SEMIPARAMETRIC ESTIMATION

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ABSTRACT. The conventional Cobb-Douglas production function model requires strong assumptions such as the constant elasticity and the entire frontier efficiency. The varying elasticity production function is a useful extension but it does not take frontier efficiency into account, and the data envelope analysis model involves the use of linear programming methods to estimate the frontier efficiency whereas it is being short of production function forms. In this paper, we combine the advantages of the varying elasticity production function and the data envelope analysis models, and put forwards semi-parametric varying coefficient estimation for the frontier varying elasticity. Empirical results on China’s provinces show that the frontier varying elasticity is significant different from the classical one, and frontier efficiency may enhances the capital elasticity and reduces the labor elasticity simultaneously.

Keywords: Production Function; Semi-Parametric Estimation; Data Envelope Analysis; Varying Elasticity

1. Introduction. Production functions are a fundamental component of all economics. Production functions relate productive inputs (e.g., capital, labor) to outputs, reflect the effect and influence of production factor on output at certain technological conditions. Cobb-Douglas production function is preferred for its simple structure, meaningful parameters and easy estimation. Assuming two factors production, physical capital (K) and labor (L), the Cobb-Douglas production function is as follow:

\[ Y = AK^\alpha L^\beta \] (1)

However, the conventional economic growth model such as Cobb-Douglas production function requires unrealistically strong assumptions, such as the constant elasticity and the entire frontier efficiency.

Being a classical logarithmic linear model, the parameters \( \alpha \) and \( \beta \) are fixed for the Cobb-Douglas production function model (1). Varying coefficient models are useful
extension of classical linear models. They arise naturally when one wishes to examine how regression coefficients change over different groups characterized by certain covariates such as age. The potential of such a modeling technique got fully explored by the seminal work (Cleveland et al., 1991; Hastie and Tibshirani, 1993; Fan and Zhang, 1999; Huang et al., 2002; You and Chen, 2006). Iwata et al. (2003), Xu and Wu (2007) applied a nonparametric method to estimate the varying elasticity of the capital and the labor. Ahmad (2005), Xu and Watada (2007), Zhang and Xu (2009, 2010), Luo et al. (2009) and Zhang and Gu (2010) introduces the nonparametric varying coefficients model to estimate varying output elasticity of capital and labor force.

However, the varying-elasticity production function does not take frontier efficiency into account (Zhang and Gu, 2010). Modern efficiency measurement begins with Farrell (1957) who drew upon the work of Debreu (1951) and Koopmans (1951) to define a simple measure of firm efficiency which could account for multiple inputs. Frontiers have been estimated using many different methods over the past 50 years. The two principal methods are: data envelopment analysis and stochastic frontiers analysis, which involve mathematical pr ogramming and econometric methods, respectively. Data envelopment analysis involves the use of linear programming methods to construct a non-parametric piecewise frontier over the data without restrictive assumptions, so as to be able to calculate efficiencies relative to this surface.

The varying elasticity production function does not take frontier efficiency into account, while the data envelopment analysis model is being short of the form of production function. In this paper, we combine the advantages of the varying elasticity production function and the data envelopment analysis model, as well as the advantages of the parametric model and the nonparametric model, and put forwards semi-parametric varying coefficient estimation for the frontier varying elasticity. In the following section contains a description of estimation methodology for the semi-parametric varying coefficient model. The estimation and comparison of the empirical results are given and detailed discussed in Section 3. And the final section 4 contains concluding remarks.

2. Estimation Methodology. Suppose the technology level A is denoted by exponential linear composition of the technical index \( Z_j (j=1,2,...,m) \), we can derive econometric model by taking logarithm and adding random item to model (1)

\[
\ln Y_i = \sum_{j=1}^{m} \lambda_j Z_{ji} + \alpha \ln K_i + \beta \ln L_i + \varepsilon_i, \quad i = 1,2,...,n \tag{2}
\]

Obviously, the Cobb-Douglas production function model (2) is a classical logarithmic linear model. The output elasticity of capital and labor force standing by the parameters \( \alpha \) and \( \beta \) are fixed constant:

\[
\frac{\partial \ln Y_i}{\partial \ln K_i} = \frac{\partial Y_i}{\partial K_i} \cdot \frac{K_i}{Y_i} = \alpha \quad \frac{\partial \ln Y_i}{\partial \ln L_i} = \frac{\partial Y_i}{\partial L_i} \cdot \frac{L_i}{Y_i} = \beta \tag{3}
\]

With the improvement of computing facilities over the last three decades, there has been an upsurge of interest and effort in nonparametric models as researchers have realized that parametric models are inadequate in capturing the relationship between the response variable and its associated covariates in many practical situations. Varying coefficient
models, including generalized additive models as well as dynamic generalized linear models as special cases, are linear in the regresses but their coefficients are permitted to change smoothly as function of other variables.

If the coefficients α and β in model (2) are permitted to change smoothly as function of other variables such as human capital \( H \), we could construct a semi-parametric varying elasticity production function model below:

\[
\ln Y_i = \sum_{j=1}^{m} \lambda_j Z_{ji} + \alpha(H_i) \ln K_i + \beta(H_i) \ln L_i + \varepsilon_i \quad i = 1, 2, \ldots, n
\]

The semi-parametric varying elasticity production function (4) is an useful extension to the Cobb-Douglas production function, with varying output elasticity. That is, the output elasticity of capital and labor force standing by the parameters \( \alpha(H) \) and \( \beta(H) \) are variable.

The output elasticities of capital and labor force are:

\[
\frac{\partial Y_i}{\partial K_i} \frac{K_i}{Y_i} = \frac{\partial \ln Y_i}{\partial \ln K_i} = \alpha(H_i)
\]

\[
\frac{\partial Y_i}{\partial L_i} \frac{L_i}{Y_i} = \frac{\partial \ln Y_i}{\partial \ln L_i} = \beta(H_i)
\]

Model (4) is obviously a typical semi-parametric varying-coefficients model. One should note that in model (4) we have not restricted \( \alpha(H) \) and \( \beta(H) \) to have a fixed impact on the dependent variable, and assume that the functions \( \alpha(H) \) and \( \beta(H) \) possess about the same degrees of smoothness and hence they can be approximated equally well in the same interval. Different to the former research, The cross section data is used in this paper other than time series, and the coefficients are allowed to change as the function of human capital in model (4) in this paper.

There are many approaches to estimating the unknown parameters and the varying coefficient functions, such as backfitting estimation (Hua et al., 2003), efficient estimation (Ahmad, 2005), and profile estimation (Fan and Huang, 2005). Profile least squares is a useful approach and it is utilized in this paper.

Suppose that we have a random sample of size \( n \) \( \{(Z_i, K_i, L_i, Y_i), i = 1, \ldots, n\} \). For any given \( \gamma \), (4) can be written as:

\[
\ln Y_i = \alpha(H_i) \ln K_i + \beta(H_i) \ln L_i + \varepsilon_i
\]

Where \( \ln Y_i = \ln Y_i - \sum_{i=1}^{n} \gamma_{Z_i} Z_i \), this transforms the varying-coefficient partially linear model (4) into the varying-coefficient model (4). The local linear regression technique is applied to estimate the coefficient functions, For \( H \) is in a small neighborhood of \( H_0 \), approximate the functions \( \alpha(H) \) and \( \beta(H) \) locally as:

\[
\alpha(H) \approx \alpha(H_0) + \alpha'(H_0) (H - H_0) \equiv a_0 + b_0 (H - H_0)
\]

\[
\beta(H) \approx \beta(H_0) + \beta'(H_0) (H - H_0) \equiv a_1 + b_1 (H - H_0)
\]

Denote \( X_{1i} = \ln K_i, \quad X_{2i} = \ln L_i \). This leads to the following weighted local least-squares problem, find \((a_0, b_0), (a_1, b_1)\) so as to minimize

\[
\text{Min} \sum_{i=1}^{n} \left( \ln Y_i^* - \sum_{j=0}^{1} (a_j + b_j (H_i - H_0) X_{ji}) \right)^2 K_i
\]
Where $K$ is a kernel function, $h$ is a bandwidth and $K_h = K(\bullet/h)/h$. We give more weight to contributions from observations very close to than to those coming from observations that are more distant. We choose the kernel function of Gaussian and choose Bandwidth $h$ with the method of cross-validation in this paper.

Denote

\[
Y = (Y_1, \ldots, Y_n)^T, \quad Z = (Z_1, \ldots, Z_n)^T, \quad Z_i = (Z_{i1}, \ldots, Z_{im})^T \\
X = (X_1, \ldots, X_n)^T, X_i = (X_{i1}, X_{i2})^T, \quad a(H) = (\alpha(H), \beta(H))^T \\
W = \text{diag}(K_h(H_1 - H), \ldots, K_h(H_n - H))
\]

\[
M = \begin{pmatrix}
a^T(H)X_1 \\
\vdots \\
a^T(H)X_n
\end{pmatrix}, \quad D = \begin{pmatrix}
X_1^T \frac{H_1 - H}{h}X_1^T \\
\vdots \\
X_n^T \frac{H_n - H}{h}X_n^T
\end{pmatrix}
\]

Then (4) can be written as

\[
Y - Z\gamma = M + \varepsilon \tag{11}
\]

The solution for $M$ is:

\[
\hat{M} = \begin{pmatrix}
(X^T 0)\{D^T WD\}^{-1}D^T W \\
\vdots \\
(X^T 0)\{D^T WD\}^{-1}D^T W
\end{pmatrix}(Y - Z\gamma) = S(Y - Z\gamma) \tag{12}
\]

The matrix $S$ is a smoothing matrix, substituting (12) into (11), applying least squares to the linear model, we obtain

\[
\hat{\gamma} = \{Z^T(I - S)^T(I - S)Z\}^{-1}Z^T(I - S)(I - S)Y \tag{13}
\]

\[
\hat{M} = S(Y - Z\hat{\gamma}) \tag{14}
\]

The solution to the problem (10) is given by:

\[
(a_1^\ast, h_1^\ast, h_2) = \{D^T WD\}^{-1}D^T W(Y - Z\hat{\gamma}) \tag{15}
\]

3. Data Description and Empirical Results

3.1. Data Description. The main variables contain Gross Domestic product ($Y$), Capital ($K$), Labour force ($L$), Human Capital ($H$), and Economic Structure ($Z$). We choose the 30 provinces of China at the year 2003 and the year 2008. In order to eliminate the influence of inflation, we calculate the true data on the base year of 1952. Gross Domestic Product, which stands for output in the paper, is calculated by expenditure approach. The number of labour force is calculated by total employed persons at the year-end. In this paper, we follow Zhang (2004) to measure the capital, and follow Tang (2006) to measure the human capital. The Economic Structure is defined as the ratio of labour force in the third industry to the whole country.

3.2. Frontier Efficiency Estimation with DEA. Efficiency measurement has been a subject of tremendous interest as organizations have struggled to improve productivity. Reasons for this focus were best stated fifty years ago by Farrell (1957) in his classic paper on the measurement of productive efficiency. Twenty years after Farrell’s seminal work, and building on those ideas, Charnes et al. (1978), responding to the need for satisfactory
procedures to assess the relative efficiencies of multi-input multi-output production units, introduced a powerful methodology which has subsequently been titled data envelopment analysis (DEA). Since the advent of data envelopment analysis in 1978, there has been an impressive growth both in theoretical developments and applications of the ideas to practical situations. Banker et al. (1984) (BCC), extended the earlier work of Charnes et al. (1978) by providing for variable returns to scale (VRS). Wade et al. (2009) provide a sketch of some of the major research thrusts in data envelopment analysis over the past three decades.

The CRS assumption is only appropriate when all DMU’s are operating at an optimal scale (i.e., one corresponding to the flat portion of the LRAC curve). Imperfect competition, constraints on finance, etc. may cause a DMU to be not operating at optimal scale. Banker, Charnes and Cooper (1984) suggested an extension of the CRS DEA model to account for variable returns to scale (VRS) situations. The use of the CRS specification when not all DMU’s are operating at the optimal scale will result in measures of TE which are confounded by scale efficiencies (SE). The use of the VRS specification will permit the calculation of TE devoid of these SE effects.

The VRS linear programming problem can be provided as:
\[
\begin{align*}
\text{Max} & \quad \theta_1 \theta, \quad s.t. \quad y_i + \lambda x_i & \geq 0, \quad \theta \geq 0, \quad \lambda \geq 0 \\
& \quad \lambda = 1, \lambda \geq 0
\end{align*}
\]

Where \( N1 \) is a \( N \times 1 \) vector of ones. This approach forms a convex hull of intersecting planes which enforces the data points more tightly than the CRS conical hull and thus provides technical efficiency scores which are greater than or equal to those obtained using the CRS model. The VRS specification has been the most commonly used specification in the 1990’s.

Many studies have decomposed the TE scores obtained from a CRS data envelopment analysis into two components, one due to scale inefficiency and one due to “pure” technical inefficiency. This may be done by conducting both a CRS and a VRS DEA upon the same data. If there is a difference in the two TE scores for a particular DMU, then this indicates that the DMU has scale inefficiency, and that the scale inefficiency can be calculated from the difference between the VRS TE score and the CRS TE score.

For the convenience of comparison, we use the CRS model to calculate the frontier efficiency in this paper, with output of GDP (lnY) and inputs of capital (lnK) and labor force (lnL). The results of the efficiencies are shown in Table 1: (1) The difference of frontier efficiency for the year 2003 and 2008 is gentle. (2) Four provinces, Liaoning, Shanghai, Tianjin and Yunnan, are under the entire efficiency 1. (3) Five provinces, Anhui, Fujian, Hubei, Jiangsu and Zhejiang, have a higher efficiency between 0.9 and 1.0. (4) and the residual provinces have average efficiency between 0.8 and 0.9, without Qinghai and Xinjiang, which have efficiency less than 0.8.

3.3. Frontier Elasticity. Varying elasticity production function model allows the variety of the elasticity of output to vary over different human capital levels. Is there any difference between normal elasticity and frontier elasticity? In this paper, we set benchmark model and frontier model to compare the varying elasticity production function and frontiers varying elasticity production function. We adding the restriction \( \alpha(H) + \beta(H) = 1 \) to model (4) for the convenience of comparison.
<table>
<thead>
<tr>
<th>Province</th>
<th>Frontier Efficiency</th>
<th>Elasticity (Benchmark Model $A_0$)</th>
<th>Frontier Elasticity (Frontier Model $A_1$)</th>
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<tr>
<td></td>
<td></td>
<td>$a_0$</td>
<td>$1-a_0$</td>
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<td>Jiangxi</td>
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<td>0.895</td>
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</table>
Benchmark Model $A_0$: Semi-parameter varying elasticity production function

$$y_i = \sum_{j=1}^{m} \lambda_{ij} Z_{ij} + \alpha_i (H_i) k_i + [1 - \alpha_i (H_i)] y_i + \varepsilon_i$$  \hspace{1cm} (17)$$

Frontier Model $A_1$: Frontier Semi-parameter varying elasticity production function

$$fy_i = \sum_{j=1}^{m} \lambda_{ij} Z_{ij} + \alpha_i (H_i) k_i + [1 - \alpha_i (H_i)] y_i + \varepsilon_i$$  \hspace{1cm} (18)$$

With $y = \ln(Y), k = \ln(K), l = \ln(L)$ and $fy = y / fe$, $fe$ is the frontier efficiency estimated by data envelopment analysis model.

Suppose the linear structure is defined by constant ($Z1$) and economic structure ($Z2$). The detail results for models $A_0$ and $A_1$ are displayed in Table 1. The comparisons for the elasticity of capital and labor force according to the semi-parameter varying elasticity production function and the frontier semi-parameter varying elasticity production function are explicitly showed in Figures 1-8.

As the figures 1-8 indicate, the varying elasticity as well as the frontier varying elasticity of capital performs an inverted U-shape trend, and the varying elasticity as well as the frontier varying elasticity of labor force performs a U-shape trend. Kuznets (1955) hypothesizes that the income inequality first increases and then decreases, thereby forming an inverted U-shaped curve. Li et.al (2009) explain the U-shape of labor share in the initial distribution. Under entirety competition and constant return to scales, the elasticity of labor force is equal to labor share, the U-shape of the varying elasticity of labour force positively support the results of Li et.al (2009). How to make the economic growth and development process more equitable and let the fruits of growth be shared more extensively among all social classes has become a key point of attention in devising the socioeconomic development strategy. These are the issues warranting further study (Xu and Zhang, 2010).

As can be seen from the Figures 5-8, the results for the frontier varying elasticity production function are differently from that of the varying elasticity production function. As the rising of human capital, the frontier elasticity of capital is bigger than the normal one, and the frontier elasticity of labour force is less than the normal one. The results indicate that the missing of frontier efficiency will underestimate the varying elasticity of capital, and
will overestimate the varying elasticity of labour force. In other words, the frontier efficiency may enhance the varying elasticity of capital and reduces the varying elasticity of labour force simultaneously.


Cobb-Douglas production function is preferred for its simple structure, meaningful parameter and easy estimation. However, the conventional Cobb-Douglas production function requires unrealistically strong assumptions, such as the constant elasticity and the full efficiency of the technique.

With the improvement of computing facilities over the last three decades, there has been an upsurge of interest and effort in nonparametric models. Varying coefficient models are a useful extension of classical linear models. They arise naturally when one wishes to examine how regression coefficients change over different groups characterized by certain covariates. The varying elasticity production function is an useful extension but it does not take frontier efficiency into account. and the data envelope analysis model involves the use of linear programming methods to estimate the frontier efficiency without restrictive assumptions whereas it is being short of production function forms.

In this paper, we combine the advantages of the varying elasticity production function and the data envelopment analysis model, as well as the parametric model and the nonparametric model, and put forwards semi-parametric varying coefficient estimation for the frontier varying elasticity.

Empirical results on China’s provinces show the following conclusions: (1) The difference of frontier efficiency for the year 2003 and 2008 is gentle. Four provinces, Liaoning, Shanghai, Tianjin and Yunnan, are under the entire efficiency 1. Five provinces, Anhui, Fujian, Hubei, Jiangsu and Zhejiang, have a higher efficiency between 0.9 and 1.0. and the residual provinces have average efficiency between 0.8 and 0.9, without Qinghai and Xinjiang, which have efficiency less than 0.8. (2) the varying elasticity as well as the frontier varying elasticity of capital performs an inverted U-shape trend, and the varying elasticity as well as the frontier varying elasticity of labour force performs an U-shape trend. Under entire competition and constant return to scales, the elasticity of labor force is equal to labor share, the U-shape of the varying elasticity of labour force positively support the results of the U-shape of labor share in the initial distribution by Li et.al (2009). (3) As the rising of human capital, the frontier elasticity of capital is bigger than the normal one, and the frontier elasticity of labour force is less the normal one. The results indicate that, the missing of frontier efficiency will underestimate the varying elasticity of capital, and will overestimate the varying elasticity of labour force. In other words, the frontier efficiency may enhance the varying elasticity of capital and reduces the varying elasticity of labour force simultaneously.

Acknowledgment. This work is partially supported by Humanities and Social Science Research of The Ministry of Education of China (NO.10YJC790065), National Philosophy and Social Science Foundation of China (NO.09CTJ005) and The Foundation of Zhejiang Gongshang University. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers.
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